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ON THE A.E. CONVERGENCE OF THE ARITHMETIC MEANS OF DOUBLE ORTHOGONAL SERIES

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ABSTRACT. The extension of the coefficient test of Menšov and Kaczmarz ensuring the a.e. $(C, 1, 1)$-summability of double orthogonal series has been stated by two authors. Unfortunately, their proofs turned out to be deficient. Now we present a general theory, in the framework of which a complete proof of this test can also be obtained. Besides, we extend the relevant theorems of Kolmogorov and Kaczmarz from single orthogonal series to double ones, establishing the a.e. equiconvergence of the lacunary subsequences of the rectangular partial sums and of the entire sequence of the arithmetic means. The corresponding tests ensuring the a.e. $(C, 1, 0)$ and $(C, 0, 1)$-summability are also treated.

1. Introduction. Let $(X, \mathcal{F}, \mu)$ be a positive measure space and $\{\phi_{ik}(x)\colon i, k = 1, 2, \ldots\}$ an orthonormal system on X. We will consider the double orthogonal series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}\phi_{ik}(x), \tag{1.1}$$

where $\{a_{ik}\colon i, k = 1, 2, \ldots\}$ is a sequence of coefficients for which

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 < \infty. \tag{1.2}$$

By the Riesz-Fischer theorem there exists a function $f(x) \in L^2 = L^2(X, \mathcal{F}, \mu)$ such that series (1.1) is the Fourier series of $f(x)$ with respect to the system $\{\phi_{ik}(x)\}$ and the rectangular partial sums

$$s_{mn}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik}\phi_{ik}(x) \qquad (m, n = 1, 2, \ldots)$$

converge to $f(x)$ in the L^2-metric:

$$\int [s_{mn}(x) - f(x)]^2 \, d\mu(x) \to 0 \quad \text{as } \min(m, n) \to \infty.$$

Here and in the sequel, the integrals are taken over the entire space X.

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It is well known that condition (1.2) does not ensure the pointwise convergence of $s_{mn}(x)$ to $f(x)$. The extension of the Rademacher-Menšov theorem proved by a number of authors (see, e.g. [**1**, **11**] etc.) reads as follows.

THEOREM A. *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log(i+1)]^2[\log(k+1)]^2 < \infty,$$

then

$$s_{mn}(x) \to f(x) \ \text{a.e.} \quad \text{as } \min(m,n) \to \infty$$

and there exists a function $F(x) \in L^2$ such that

$$\sup_{m,n\geq 1} |s_{mn}(x)| \leq F(x) \quad \text{a.e.}$$

Inspired by this statement, we make the following convention. Given a double sequence $\{f_{mn}(x)\}$ in functions of L^2, we write

$$f_{mn}(x) = o_x\{1\} \ \text{a.e.} \quad \text{as } \min(m,n) \to \infty$$

(or $\max(m,n) \to \infty$, or $m \to \infty$, or $n \to \infty$) if

$$f_{mn}(x) \to 0 \ \text{a.e.} \quad \text{as } \min(m,n) \to \infty$$

(or $\max(m,n) \to \infty$, or $m \to \infty$, or $n \to \infty$) and, in addition, there exists a function $F(x) \in L^2$ such that

$$\sup_{m,n} |f_{mn}(x)| \leq F(x) \quad \text{a.e.}$$

Here m ranges over either $0, 1, \ldots$ or $1, 2, \ldots$; and so does n.

The following corollaries of Theorem A are interesting in themselves.

COROLLARY A. *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log\log(i+3)]^2[\log(k+1)]^2 < \infty, \tag{1.3}$$

then

$$s_{2^p,n}(x) - f(x) = o_x\{1\} \ \text{a.e.} \quad \text{as } \min(p,n) \to \infty;$$

while if

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log(i+1)]^2[\log\log(k+3)]^2 < \infty, \tag{1.4}$$

then

$$s_{m,2^q}(x) - f(x) = o_x\{1\} \ \text{a.e.} \quad \text{as } \min(m,q) \to \infty.$$

COROLLARY B. *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log\log(i+3)]^2[\log\log(k+3)]^2 < \infty, \tag{1.5}$$

then

$$s_{2^p,2^q}(x) - f(x) = o_x\{1\} \ \text{a.e.} \quad \text{as } \min(p,q) \to \infty.$$

In this paper, the logarithms are to the base 2.

2. Main results. We will consider the first arithmetic means of the rectangular partial sums defined by

$$\sigma_{mn}(x) = \sigma_{mn}^{11}(x) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n} s_{ik}(x)$$
$$= \sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)\left(1-\frac{k-1}{n}\right) a_{ik}\phi_{ik}(x) \qquad (m,n=1,2,\ldots).$$

Besides, we will consider the arithmetic means with respect to only m:

$$\sigma_{mn}^{10}(x) = \frac{1}{m}\sum_{i=1}^{m} s_{in}(x) = \sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right) a_{ik}\phi_{ik}(x),$$

and those with respect to only n:

$$\sigma_{mn}^{01}(x) = \frac{1}{n}\sum_{k=1}^{n} s_{mk}(x) = \sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{k-1}{n}\right) a_{ik}\phi_{ik}(x).$$

The following two theorems are Kolmogorov type results for double orthogonal series (cf. [**8**] and also [**2**, pp. 118–119] concerning single orthogonal series).

THEOREM 1. *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log(k+1)]^2 < \infty, \tag{2.1}$$

then

$$s_{2^p,n}(x) - \sigma_{2^p,n}^{10}(x) = o_x\{1\} \text{ a.e. } \quad \text{as } p \to \infty \tag{2.2}$$

uniformly in n; *while if*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log(i+1)]^2 < \infty, \tag{2.3}$$

then

$$s_{m,2^q}(x) - \sigma_{m,2^q}^{01}(x) = o_x\{1\} \text{ a.e. } \quad \text{as } q \to \infty \tag{2.4}$$

uniformly in m.

THEOREM 2. *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log\log(\max(i,k)+3)]^2 < \infty, \tag{2.5}$$

then

$$s_{2^p,2^q}(x) - \sigma_{2^p,2^q}(x) = o_x\{1\} \text{ a.e. } \quad \text{as } \min(p,q) \to \infty. \tag{2.6}$$

The next two theorems are Kaczmarz type results for double orthogonal series (cf. [**6**] and also [**2**, pp. 119–120] about single orthogonal series).

THEOREM 3. *Under condition* (2.1),

$$\max_{2^p<m\le 2^{p+1}} |\sigma^{10}_{mn}(x)-\sigma^{10}_{2^p,n}(x)| = o_x\{1\} \text{ a.e. as } p\to\infty \tag{2.7}$$

uniformly in n; while under condition (2.3),

$$\max_{2^q<n\le 2^{q+1}} |\sigma^{01}_{mn}(x)-\sigma^{01}_{m,2^q}(x)| = o_x\{1\} \text{ a.e. as } q\to\infty \tag{2.8}$$

uniformly in m.

THEOREM 4. *Under condition* (2.5),

$$\max_{2^p\le m\le 2^{p+1}} \max_{2^q\le n\le 2^{q+1}} |\sigma_{mn}(x)-\sigma_{2^p,2^q}(x)| = o_x\{1\} \text{ a.e. as } \min(p,q)\to\infty. \tag{2.9}$$

Combining Theorems 1 and 3 as well as Theorems 2 and 4 we obtain the following two corollaries.

COROLLARY 1. *Under condition* (2.1), *series* (1.1) *is* $(C,1,0)$*-summable a.e. on a measurable subset* Y *of* X *if and only if* $\{s_{2^p,n}(x)\}$ *converges a.e. on* Y *as* $\min(p,n)\to\infty$; *while under condition* (2.3), *series* (1.1) *is* $(C,0,1)$*-summable a.e. on* Y *if and only if* $\{s_{m,2^q}(x)\}$ *converges a.e. on* Y *as* $\min(m,q)\to\infty$.

COROLLARY 2. *Under condition* (2.5), *series* (1.1) *is* $(C,1,1)$*-summable a.e. on a measurable subset* Y *of* X *if and only if* $\{s_{2^p,2^q}(x)\}$ *converges a.e. on* Y *as* $\min(p,q)\to\infty$.

Combining Corollary A and Corollary 1 we get a coefficient test for the a.e. $(C,1,0)$ and $(C,0,1)$-summability, respectively.

COROLLARY 3. *Under condition* (1.3),

$$\sigma^{10}_{mn}(x)-f(x)=o_x\{1\} \text{ a.e. as } \min(m,n)\to\infty;$$

while under condition (1.4),

$$\sigma^{01}_{mn}(x)-f(x)=o_x\{1\} \text{ a.e. as } \min(m,n)\to\infty.$$

Finally, combining Corollary B and Corollary 2 yields a coefficient test for the a.e. $(C,1,1)$-summability.

COROLLARY 4. *Under condition* (1.5),

$$\sigma_{mn}(x)-f(x)=o_x\{1\} \text{ a.e. as } \min(m,n)\to\infty.$$

The last two results can be considered as the extensions of the coefficient test of Menšov [**9**] and Kaczmarz [**7**] from single orthogonal series to double ones. (See also [**2**, pp. 125–126].)

3. Remarks and comments. (i) As in the case of single orthogonal series, Theorems 1 and 2 remain true if in the statements (2.2), (2.4) and (2.6) the subsequences $\{2^p\}$ and $\{2^q\}$ are replaced by $\{\kappa_p\}$ and $\{\lambda_q\}$, respectively, where $\{\kappa_p\}$ and $\{\lambda_q\}$ are lacunary sequences of positive integers in the sense of Hadamard, i.e. there exist constants u, p_0 and q_0 such that

$$1<u\le\frac{\kappa_{p+1}}{\kappa_p}\quad (p\ge p_0) \qquad\text{and}\qquad 1<u\le\frac{\lambda_{q+1}}{\lambda_q}\quad (q\ge q_0) \tag{3.1}$$

(cf. [**2**, p. 118]).

(ii) Similarly, Theorems 3 and 4 remain true if in the statements (2.7)–(2.9) the subsequences are replaced by $\{\kappa_p\}$ and $\{\lambda_q\}$, respectively, where $\{\kappa_p\}$ and $\{\lambda_q\}$ are sequences of positive integers for which there exist constants v, p_1 and q_1 such that

$$(3.2) \qquad \frac{\kappa_{p+1}}{\kappa_p} \le v < \infty \quad (p \ge p_1) \qquad \text{and} \qquad \frac{\lambda_{q+1}}{\lambda_q} \le v < \infty \quad (q \ge q_1)$$

(cf. [**2**, p. 119]).

(iii) Consequently, Corollaries 1 and 2 remain also true if the subsequences $\{2^p\}$ and $\{2^q\}$ are replaced by sequences $\{\kappa_p\}$ and $\{\lambda_q\}$ of positive integers for which both (3.1) and (3.2) are satisfied.

(iv) Corollary 4 was firstly stated by Fedulov [**4**]. Unfortunately, there are two essential defects in his proof. First, a false statement which says that, merely under condition (1.2),

$$(3.3) \qquad s_{2^p,2^q}(x) - \sigma_{2^p,2^q}(x) \to 0 \text{ a.e.} \quad \text{as } \min(p,q) \to \infty$$

was "proved" in [**4**, pp. 436–437]. Csernyák [**3**] gave a counterexample disproving (3.3) under condition (1.2), and in the same paper he proved, under condition (1.5), relation (3.3) does hold.

(v) The lack of a statement of type (2.9) is the second defect in Fedulov's proof attempt in [**4**, pp. 437–438]. Instead, he proves (2.7) for every fixed n, (2.8) for every fixed m, and (7.1) below. But the a.e. equiconvergence of the sequences $\{\sigma_{2^p,2^q}(x)\}$ and $\{\sigma_{mn}(x)\}$ does not follow from these three statements. This circumstance escaped the attention of Csernyák [**3**] who takes this unproven statement for granted.

(vi) The convergence notion used in the above theorems and corollaries is the so-called convergence in Pringsheim's sense (see, e.g. [**12**, p. 303 or **11**]). But Theorem A, Corollaries A and B are true (cf. [**11**]), and Corollaries 3 and 4 remain true if this convergence notion is replaced in them by regular convergence. The latter convergence notion was introduced by Hardy [**5**] and for a double sequence $\{f_{mn}\}$ it requires the fulfillment of the following two statements:

(a) $\{f_{mn}\}$ as a double sequence converges in Pringsheim's sense;

(b) for each fixed n, $\{f_{mn}\}$ as a single sequence (in m) converges, and for each fixed m, $\{f_{mn}\}$ as a single sequence (in n) converges.

This kind of convergence was rediscovered by the present author in [**11**], introducing the notion of convergence in a restricted sense whose definition turned out to be equivalent to the notion of regular convergence.

For instance, we show that in Corollary 4 we can state the regular convergence of $\{\sigma_{mn}(x)\}$ a.e. Statement (a) is satisfied a.e., this is the conclusion of Corollary 4. To check the a.e. fulfillment of statement (b), let us fix n say. We can rewrite

$$\sigma_{mn}(x) = \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) {}^{k}\sigma_m(x)$$

where

$${}^{k}\sigma_m(x) = \sum_{i=1}^{m} \left(1 - \frac{i-1}{m}\right) a_{ik}\phi_{ik}(x)$$

is the mth arithmetic mean of the single orthogonal series $\sum_{i=1}^{\infty} a_{ik}\phi_{ik}(x)$ (the so-called kth row of series (1.1)). By (1.5), for each k

$$\sum_{i=1}^{\infty} a_{ik}^2[\log\log(i+3)]^2 < \infty$$

whence, applying the Menšov-Kaczmarz theorem, we can conclude the a.e. convergence of $\{{}^k\sigma_m(x)\}$ as $m \to \infty$ for each k, a fortiori the a.e. convergence of $\{\sigma_{mn}(x)\}$ as $m \to \infty$ for each n. An analogous conclusion can be drawn when m is fixed. Thus, we have established the a.e. regular convergence of $\{\sigma_{mn}(x)\}$ under condition (1.5).

4. Proof of Theorem 1. It is enough to prove the first statement, i.e. (2.2), since the second statement can be proved in a similar fashion.

By definition,

$$s_{2^p,n}(x) - \sigma_{2^p,n}^{10}(x) = \sum_{i=2}^{2^p}\sum_{k=1}^{n} \frac{i-1}{2^p} a_{ik}\phi_{ik}(x) \qquad (p, n = 1, 2, \ldots).$$

(For $p = 0$ we have $s_{1n}(x) = \sigma_{1n}^{10}(x)$.) We accomplish the proof in two steps.

Step 1. First we treat the special case $n = 2^q$ ($q = 0, 1, \ldots$) and prove

$$s_{2^p,2^q}(x) - \sigma_{2^p,2^q}^{10}(x) = o_x\{1\} \text{ a.e.} \quad \text{as } p \to \infty \tag{4.1}$$

uniformly in q. To this end, by the Cauchy inequality,

$$\begin{aligned}
|s_{2^p,2^q}(x) - \sigma_{2^p,2^q}^{10}(x)| &= \left|\sum_{i=2}^{2^p}\sum_{k=1}^{2^q} \frac{i-1}{2^p} a_{ik}\phi_{ik}(x)\right| \\
&\le \sum_{r=0}^{q}\left|\sum_{i=2}^{2^p}\sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{2^p} a_{ik}\phi_{ik}(x)\right| \\
&\le \left\{\sum_{r=0}^{q}(r+1)^2\left[\sum_{i=2}^{2^p}\sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{2^p} a_{ik}\phi_{ik}(x)\right]^2\right\}^{1/2} \\
&\quad \times \left\{\sum_{r=0}^{q}\frac{1}{(r+1)^2}\right\}^{1/2}
\end{aligned}$$

with the agreement that by 2^{-1} we mean 0 in this paper. The second factor on the right does not exceed $\pi/\sqrt{6}$ for $q = 0, 1, \ldots$. Consequently, it is enough to deal with the first factor on the right.

We can estimate it in the following way: for each p and q,

$$\left\{\sum_{r=0}^{q}(r+1)^2\left[\sum_{i=2}^{2^p}\sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{2^p} a_{ik}\phi_{ik}(x)\right]^2\right\}^{1/2} \le F_1(x)$$

where

$$F_1(x) = \left\{\sum_{p=1}^{\infty}\sum_{r=0}^{\infty}(r+1)^2\left[\sum_{i=2}^{2^p}\sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{2^p} a_{ik}\phi_{ik}(x)\right]^2\right\}^{1/2}.$$

By (2.1),

$$\int F_1^2(x)\,d\mu(x) = \sum_{p=1}^{\infty}\sum_{r=0}^{\infty}(r+1)^2\sum_{i=2}^{2^p}\sum_{k=2^{r-1}+1}^{2^r}\frac{(i-1)^2}{2^{2p}}a_{ik}^2 \tag{4.2}$$

$$\le \sum_{p=1}^{\infty}\sum_{r=0}^{\infty}\sum_{i=2}^{2^p}\sum_{k=2^{r-1}+1}^{2^r}\frac{(i-1)^2}{2^{2p}}a_{ik}^2[\log 4k]^2$$

$$= \sum_{p=1}^{\infty}\sum_{k=1}^{\infty}\sum_{i=2}^{2^p}\frac{(i-1)^2}{2^{2p}}a_{ik}^2[\log 4k]^2 g$$

$$= \sum_{i=2}^{\infty}\sum_{k=1}^{\infty}(i-1)^2 a_{ik}^2[\log 4k]^2 \sum_{p:\ 2^p\ge i}\frac{1}{2^{2p}}$$

$$\le C_1\sum_{i=2}^{\infty}\sum_{k=1}^{\infty}a_{ik}^2[\log 4k]^2 < \infty,$$

where by $C_1, C_2, \ldots$ we denote positive absolute constants. Hence B. Levi's theorem (i.e. the term-by-term integration of a series with nonnegative terms, in other words, the dominated convergence theorem; see. e.g. [**13**, pp. 35–36]) implies (4.1).

Step 2. Let $2^q < n \le 2^{q+1}$ for some $q \ge 1$. Then clearly

$$\left|\sum_{i=2}^{2^p}\sum_{k=1}^{n}\frac{i-1}{2^p}a_{ik}\phi_{ik}(x)\right| \le \left|\sum_{i=2}^{2^p}\sum_{k=1}^{2^q}\frac{i-1}{2^p}a_{ik}\phi_{ik}(x)\right| + \left|\sum_{i=2}^{2^p}\sum_{k=2^q+1}^{n}\frac{i-1}{2^p}a_{ik}\phi_{ik}(x)\right|$$

whence

$$\max_{2^q<n\le 2^{q+1}}|s_{2^p,n}(x)-\sigma_{2^p,n}^{10}(x)| \le |s_{2^p,2^q}(x)-\sigma_{2^p,2^q}^{10}(x)| + M_{pq}^{(1)}(x) \tag{4.3}$$

where

$$M_{pq}^{(1)}(x) = \max_{2^q<n\le 2^{q+1}}\left|\sum_{i=2}^{2^p}\sum_{k=2^q+1}^{n}\frac{i-1}{2^p}a_{ik}\phi_{ik}(x)\right|.$$

Now we apply the Rademacher-Menšov inequality (see, e.g. [**2**, p. 79 or **10**, Theorem 3]):

$$\int [M_{pq}^{(1)}(x)]^2\,d\mu(x) \le [\log 2^{q+1}]^2\sum_{i=2}^{2^p}\sum_{k=2^q+1}^{2^{q+1}}\frac{(i-1)^2}{2^{2p}}a_{ik}^2$$

$$\le \sum_{i=2}^{2^p}\sum_{k=2^q+1}^{2^{q+1}}\frac{(i-1)^2}{2^{2p}}a_{ik}^2[\log 2k]^2.$$

Setting

$$F_2(x) = \left\{\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}[M_{pq}^{(1)}(x)]^2\right\}^{1/2}$$

we get, in the same manner as in (4.2), that

$$\int F_2^2(x)\,d\mu(x) \le \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{i=2}^{2^p}\sum_{k=2^q+1}^{2^{q+1}} \frac{(i-1)^2}{2^{2p}} a_{ik}^2 [\log 2k]^2$$
$$= \sum_{p=1}^{\infty}\sum_{k=3}^{\infty}\sum_{i=2}^{2^p} \frac{(i-1)^2}{2^{2p}} a_{ik}^2 [\log 2k]^2 < \infty.$$

Hence B. Levi's theorem implies

(4.4) $$M_{pq}^{(1)}(x) = o_x\{1\} \text{ a.e. } \quad \text{as } \max(p,q) \to \infty.$$

Combining (4.1), (4.3) and (4.4), we find (2.2) to be proved.

5. Proof of Theorem 2. Clearly,

$$s_{2^p,2^q}(x) - \sigma_{2^p,2^q}(x) = \sum_{i=1}^{2^p}\sum_{k=1}^{2^q} \left(\frac{i-1}{2^p} + \frac{k-1}{2^q} - \frac{(i-1)(k-1)}{2^p 2^q} \right) a_{ik}\phi_{ik}(x)$$
$$= [s_{2^p,2^q}(x) - \sigma_{2^p,2^q}^{10}(x)] + [s_{2^p,2^q}(x) - \sigma_{2^p,2^q}^{01}(x)]$$
$$- \sum_{i=2}^{2^p}\sum_{k=2}^{2^q} \frac{(i-1)(k-1)}{2^p 2^q} a_{ik}\phi_{ik}(x).$$

Accordingly, we divide the proof of (2.6) into three parts.

Part 1. First we prove that if

(5.1) $$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 [\log\log(k+3)]^2 < \infty,$$

then

(5.2) $$s_{2^p,2^q}(x) - \sigma_{2^p,2^q}^{10}(x) = o_x\{1\} \text{ a.e. } \quad \text{as } p \to \infty$$

uniformly in q.

This statement is a consequence of Theorem 1. In fact, setting

$$a_{ir}^* = \left\{ \sum_{k=2^{r-1}+1}^{2^r} a_{ik}^2 \right\}^{1/2} \qquad (r = 0, 1, \ldots)$$

and

$$\phi_{ir}^*(x) = \begin{cases} \dfrac{1}{a_{ir}^*} \left\{ \displaystyle\sum_{k=2^{r-1}+1}^{2^r} a_{ik}\phi_{ik}(x) \right\}^{1/2} & \text{if } a_{ir}^* \ne 0, \\ \phi_{i,2^r}(x) & \text{if } a_{ir}^* = 0; \end{cases}$$

we obtain a new orthonormal system $\{\phi_{ir}^*(x)\colon i = 1, 2, \ldots;\ r = 0, 1, \ldots\}$. By (5.1),

$$\sum_{i=1}^{\infty}\sum_{r=0}^{\infty} [a_{ir}^*]^2 [\log(r+2)]^2 < \infty.$$

Thus we can apply Theorem 1 and get that

(5.3) $$s_{2^p,q}^*(x) - \sigma_{2^p,q}^{*10}(x) = o_x\{1\} \text{ a.e. } \quad \text{as } p \to \infty$$

uniformly in q, where

$$s^*_{2^p,q}(x) - \sigma^{*10}_{2^p,q}(x) = \sum_{i=2}^{2^p}\sum_{r=0}^{q} \frac{i-1}{2^p} a^*_{ir}\phi^*_{ir}(x)$$
$$= \sum_{i=2}^{2^p}\sum_{k=1}^{2^q} \frac{i-1}{2^p} a_{ik}\phi_{ik}(x) = s_{2^p,2^q}(x) - \sigma^{10}_{2^p,2^q}(x).$$

That is, (5.3) is equivalent to (5.2) to be proved.

Part 2. In the same manner we can deduce that if

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a^2_{ik}[\log\log(i+3)]^2 < \infty, \tag{5.4}$$

then

$$s_{2^p,2^q}(x) - \sigma^{01}_{2^p,2^q}(x) = o_x\{1\} \text{ a.e.} \quad \text{as } q \to \infty$$

uniformly in p.

Part 3. Finally, it is easy to show that under condition (1.2),

$$A^{(2)}_{pq}(x) = \sum_{i=2}^{2^p}\sum_{k=2}^{2^q} \frac{(i-1)(k-1)}{2^p 2^q} a_{ik}\phi_{ik}(x) = o_x\{1\} \text{ a.e.} \quad \text{as } \max(p,q) \to \infty. \tag{5.5}$$

Indeed,

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \int [A^{(2)}_{pq}(x)]^2 \, d\mu(x) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{i=2}^{2^p}\sum_{k=2}^{2^q} \frac{(i-1)^2(k-1)^2}{2^{2p}2^{2q}} a^2_{ik}$$
$$= \sum_{i=2}^{\infty}\sum_{k=2}^{\infty} (i-1)^2(k-1)^2 a^2_{ik} \sum_{p:\ 2^p \geq i} \ \sum_{q:\ 2^q \geq k} \frac{1}{2^{2p}2^{2q}}$$
$$\leq C_2 \sum_{i=2}^{\infty}\sum_{k=2}^{\infty} a^2_{ik} < \infty$$

and B. Levi's theorem implies (5.5).

6. Proof of Theorem 3. As in the case of Theorem 1, it is enough to prove the first statement. Even we prove somewhat more: under condition (2.1),

$$A^{(3)}_{pn}(x) = \sum_{m=2^p+1}^{2^{p+1}} |\sigma^{10}_{mn}(x) - \sigma^{10}_{m-1,n}(x)| = o_x\{1\} \text{ a.e.} \quad \text{as } p \to \infty \tag{6.1}$$

uniformly in n. Since

$$\max_{2^p < m \leq 2^{p+1}} |\sigma^{10}_{mn}(x) - \sigma^{10}_{2^p,n}(x)| \leq A^{(3)}_{pn}(x)$$

hence (2.7) immediately follows.

We will prove again in two steps, using the representation

$$\sigma^{10}_{mn}(x) - \sigma^{10}_{m-1,n}(x) = \sum_{i=2}^{m}\sum_{k=1}^{n} \frac{i-1}{m(m-1)} a_{ik}\phi_{ik}(x) \tag{6.2}$$

$$(m = 2, 3, \ldots; n = 1, 2, \ldots).$$

Step 1. First we prove (6.1) in the special case $n = 2^q$:

$$A^{(3)}_{p,2^q}(x) = o_x\{1\} \text{ a.e.} \quad \text{as } p \to \infty \tag{6.3}$$

uniformly in q. To this effect, by (6.2) and the Cauchy inequality,

$$A^{(3)}_{p,2^q}(x) \le \sum_{m=2^p+1}^{2^{p+1}} \sum_{r=0}^{q} \left| \sum_{i=2}^{m} \sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{m(m-1)} a_{ik}\phi_{ik}(x) \right|$$

$$\le \frac{\pi}{\sqrt{6}} \left\{ \sum_{m=2^p+1}^{2^{p+1}} \sum_{r=0}^{q} m(r+1)^2 \left[\sum_{i=2}^{m} \sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{m(m-1)} a_{ik}\phi_{ik}(x) \right]^2 \right\}^{1/2}.$$

This inequality suggests defining

$$F_3(x) = \left\{ \sum_{m=2}^{\infty} \sum_{r=0}^{\infty} m(r+1)^2 \left[\sum_{i=2}^{m} \sum_{k=2^{r-1}+1}^{2^r} \frac{i-1}{m(m-1)} a_{ik}\phi_{ik}(x) \right]^2 \right\}^{1/2}.$$

By (2.1),

$$\int F_3^2(x)\, d\mu(x) = \sum_{m=2}^{\infty} \sum_{r=0}^{\infty} m(r+1)^2 \sum_{i=2}^{m} \sum_{k=2^{r-1}+1}^{2^r} \frac{(i-1)^2}{m^2(m-1)^2} a_{ik}^2 \tag{6.4}$$

$$\le \sum_{m=2}^{\infty} \sum_{r=0}^{\infty} \sum_{i=2}^{\infty} \sum_{k=2^{r-1}+1}^{2^r} \frac{i^2}{m^3} a_{ik}^2 [\log 4k]^2$$

$$= \sum_{m=2}^{\infty} \sum_{k=1}^{\infty} \sum_{i=2}^{m} \frac{i^2}{m^3} a_{ik}^2 [\log 4k]^2$$

$$= \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} i^2 a_{ik}^2 [\log 4k]^2 \sum_{m=i}^{\infty} \frac{1}{m^3}$$

$$\le C_3 \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log 4k]^2 < \infty.$$

Hence B. Levi's theorem implies (6.3).

Step 2. We proceed similarly to Step 2 in the proof of Theorem 1. Let $q \ge 1$. Then by (6.2),

$$\max_{2^q < n \le 2^{q+1}} A^{(3)}_{pn}(x) \le A^{(3)}_{p,2^q}(x) + \sum_{m=2^p+1}^{2^{p+1}} M^{(3)}_{mq}(x) \tag{6.5}$$

where

$$M^{(3)}_{mq}(x) = \max_{2^q < n \le 2^{q+1}} \left| \sum_{i=2}^{m} \sum_{k=2^q+1}^{n} \frac{i-1}{m(m-1)} a_{ik}\phi_{ik}(x) \right|.$$

Applying the Cauchy inequality:

$$\sum_{m=2^p+1}^{2^{p+1}} M^{(3)}_{mq}(x) \le \left\{ \sum_{m=2^p+1}^{2^{p+1}} m[M^{(3)}_{mq}(x)]^2 \right\}^{1/2}, \tag{6.6}$$

then the Rademacher-Menšov inequality separately for each fixed m:

$$\int [M_{mq}^{(3)}(x)]^2 \, d\mu(x) \leq [\log 2^{q+1}]^2 \sum_{i=2}^{m} \sum_{k=2^q+1}^{2^{q+1}} \frac{(i-1)^2}{m^2(m-1)^2} a_{ik}^2$$

$$\leq \sum_{i=2}^{m} \sum_{k=2^q+1}^{2^{q+1}} \frac{i^2}{m^4} a_{ik}^2 [\log 2k]^2.$$

Then setting

$$F_4(x) = \left\{ \sum_{m=2}^{\infty} \sum_{q=1}^{\infty} m[M_{mq}^{(3)}(x)]^2 \right\}^{1/2}$$

we get in the same way as in (6.4) that

$$\int F_4^2(x) \, d\mu(x) \leq \sum_{m=2}^{\infty} \sum_{q=1}^{\infty} \sum_{i=2}^{m} \sum_{k=2^q+1}^{2^{q+1}} \frac{i^2}{m^3} a_{ik}^2 [\log 2k]^2$$

$$= \sum_{m=2}^{\infty} \sum_{k=3}^{\infty} \sum_{i=2}^{m} \frac{i^2}{m^3} a_{ik}^2 [\log 2k]^2 < \infty.$$

Hence, B. Levi's theorem implies, through (6.6),

$$\sum_{m=2^p+1}^{2^{p+1}} M_{mq}^{(3)}(x) = o_x\{1\} \text{ a.e.} \quad \text{as } p \to \infty \tag{6.7}$$

uniformly in q.

Putting (6.3), (6.5) and (6.7) together, we find (6.1) to be proved.

7. Proof of Theorem 4. From the identity

$$\sigma_{mn}(x) - \sigma_{2^p,2^q}(x) = [\sigma_{mn}(x) - \sigma_{m,2^q}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^q}(x)]$$
$$+ [\sigma_{m,2^q}(x) - \sigma_{2^p,2^q}(x)] + [\sigma_{2^p,n}(x) - \sigma_{2^p,2^q}(x)]$$

it follows immediately that

$$\max_{2^p \leq m \leq 2^{p+1}} \max_{2^q \leq n \leq 2^{q+1}} |\sigma_{mn}(x) - \sigma_{2^p,2^q}(x)|$$
$$\leq \max_{2^p \leq m \leq 2^{p+1}} \max_{2^q \leq n \leq 2^{q+1}} |\sigma_{mn}(x) - \sigma_{m,2^q}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^q}(x)|$$
$$+ \max_{2^p < m \leq 2^{p+1}} |\sigma_{m,2^q}(x) - \sigma_{2^p,2^q}(x)| + \max_{2^q < n \leq 2^{q+1}} |\sigma_{2^p,n}(x) - \sigma_{2^p,2^q}(x)|$$
$$\leq \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^q+1}^{2^{q+1}} |\sigma_{mn}(x) - \sigma_{m-1,n}(x) - \sigma_{m,n-1}(x) + \sigma_{m-1,n-1}(x)|$$
$$+ \sum_{m=2^p+1}^{2^{p+1}} |\sigma_{m,2^q}(x) - \sigma_{m-1,2^q}(x)| + \sum_{n=2^q+1}^{2^{q+1}} |\sigma_{2^p,n}(x) - \sigma_{2^p,n-1}(x)|$$
$$= A_{pq}^{(4)}(x) + A_{pq}^{(5)}(x) + A_{pq}^{(6)}(x).$$

According to this estimate, the proof is made in three parts.

Part 1. It was proved by Fedulov [**4**], that under condition (1.2),

$$A_{pq}^{(4)}(x) = o_x\{1\} \text{ a.e. as } \max(p,q) \to \infty. \tag{7.1}$$

For the sake of completeness, we insert here the estimation the proof is based on:

$$A_{pq}^{(4)}(x) \le \left\{ \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^q+1}^{2^{q+1}} mn[\sigma_{mn}(x) - \sigma_{m-1,n}(x) - \sigma_{m,n-1}(x) + \sigma_{m-1,n-1}(x)]^2 \right\}^{1/2}$$

$$= \left\{ \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^q+1}^{2^{q+1}} mn \left[\sum_{i=2}^{m} \sum_{k=2}^{n} \frac{(i-1)(k-1)}{m(m-1)n(n-1)} a_{ik}\phi_{ik}(x) \right]^2 \right\}^{1/2}.$$

Part 2. We will prove that under condition (5.1),

$$A_{pq}^{(5)}(x) = o_x\{1\} \text{ a.e. as } p \to \infty \tag{7.2}$$

uniformly in q.

Using the representation

$$\sigma_{mn}(x) - \sigma_{m-1,n}(x) = \sum_{i=2}^{m} \sum_{k=1}^{n} \frac{i-1}{m(m-1)} \left(1 - \frac{k-1}{n}\right) a_{ik}\phi_{ik}(x)$$

$$(m = 2, 3, \ldots; n = 1, 2, \ldots)$$

and taking (6.2) into account we can write

$$\sigma_{m,2^q}(x) - \sigma_{m-1,2^q}(x) = [\sigma_{m,2^q}^{10}(x) - \sigma_{m-1,2^q}^{10}(x)] - \sum_{i=2}^{m} \sum_{k=2}^{2^q} \frac{(i-1)(k-1)}{m(m-1)2^q} a_{ik}\phi_{ik}(x).$$

Hence,

$$A_{pq}^{(5)}(x) \le A_{p,2^q}^{(3)}(x) + A_{pq}^{(7)}(x) \tag{7.3}$$

where $A_{pn}^{(3)}(x)$ was defined by (6.1) (now $n = 2^q$) and

$$A_{pq}^{(7)}(x) = \sum_{m=2^p+1}^{2^{p+1}} \left| \sum_{i=2}^{m} \sum_{k=2}^{2^q} \frac{(i-1)(k-1)}{m(m-1)2^q} a_{ik}\phi_{ik}(x) \right|.$$

First, using the same "contraction" technique as in the proof of Part 1 during the proof of Theorem 2, from estimate (6.1) we can deduce that under condition (5.1),

$$A_{p,2^q}^{(3)}(x) = \sum_{m=2^p+1}^{2^{p+1}} \left| \sum_{i=2}^{m} \sum_{k=1}^{2^q} \frac{i-1}{m(m-1)} a_{ik}\phi_{ik}(x) \right| = o_x\{1\} \text{ a.e. as } p \to \infty \tag{7.4}$$

uniformly in q.

Second, it is not hard to prove that under condition (1.2),

$$A_{pq}^{(7)}(x) = o_x\{1\} \text{ a.e.} \quad \text{as } \max(p,q) \to \infty. \tag{7.5}$$

In fact, by the Cauchy inequality

$$A_{pq}^{(7)}(x) \le \left\{ \sum_{m=2^p+1}^{2^{p+1}} m \left[\sum_{i=2}^{m} \sum_{k=2}^{2^q} \frac{(i-1)(k-1)}{m(m-1)2^q} a_{ik}\phi_{ik}(x) \right]^2 \right\}^{1/2}.$$

Setting

$$F_5(x) = \left\{ \sum_{m=2}^{\infty} \sum_{q=1}^{\infty} m \left[\sum_{i=2}^{m} \sum_{k=2}^{2^q} \frac{(i-1)(k-1)}{m(m-1)2^q} a_{ik}\phi_{ik}(x) \right]^2 \right\}^{1/2},$$

by (1.2),

$$\begin{aligned} \int F_5^2(x)\, d\mu(x) &= \sum_{m=2}^{\infty} \sum_{q=1}^{\infty} m \sum_{i=2}^{m} \sum_{k=2}^{2^q} \frac{(i-1)^2(k-1)^2}{m^2(m-1)^2 2^{2q}} a_{ik}^2 \\ &\le \sum_{m=2}^{\infty} \sum_{q=1}^{\infty} \sum_{i=2}^{m} \sum_{k=2}^{2^q} \frac{i^2k^2}{m^3 2^{2q}} a_{ik}^2 \\ &= \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} i^2k^2 a_{ik}^2 \sum_{m=i}^{\infty} \frac{1}{m^3} \sum_{q:\ 2^q \ge k} \frac{1}{2^{2q}} \\ &\le C_4 \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 < \infty. \end{aligned}$$

Hence B. Levi's theorem implies (7.5).

Collecting (7.3), (7.4) and (7.5) yields (7.2).

Part 3. The companion statement that under condition (5.4),

$$A_{pq}^{(6)}(x) = o_x\{1\} \text{ a.e.} \quad \text{as } q \to \infty$$

uniformly in p, can be proved similarly.

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